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Nonlinear stability of a stratified shear flow: a viscous critical layer

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The nonlinear stability of a weakly supercritical shear flow with vertical temperature (density) stratification is investigated. It is shown that the usual Lin's rule of 'indenting' a singularity at the point of wave–flow resonance (the so-called critical layer, CL) is inapplicable for evaluating the nonlinear effects. To this end, a consistent procedure for deriving a nonlinear evolution equation is suggested and realized for the viscous critical-layer regime. The procedure takes into account the interaction of the fundamental harmonic with the second harmonic as well as with the zeroth one (i.e. with the mean-flow distortion). It is shown that the nonlinear factors both act in the same manner – at Prandtl number $\eta < 1$ they limit the instability but at $\eta > 1$ they enhance it and convey a 'burst-like' character to it.

It is found that CL is the region of strongest interactions between the harmonics. Hence the nonlinear contribution does not actually depend on the type of original flow model chosen. A simple physical interpretation is given to illustrate the mechanism governing the nonlinearity effects on the stability in the viscous critical-layer regime.

1. Introduction

The instability of shear flows is extensively invoked for explaining various phenomena in hydrodynamics, the physics of the atmosphere and ocean, geophysics, and so on. A linear stability theory for such flows in a uniform (Lin 1955) as well as a stratified fluid (Drazin & Howard 1966; Gossard & Hooke 1975) has, in general, now been constructed. One of the central concepts underlying this theory is that of critical layers (CL), i.e. the neighbourhoods of the surfaces $y = y_c$, where the phase velocity c of a neutral perturbation coincides with the flow velocity $u_x = u(y)$. Depending on which of the three scales

$$l_t = \frac{\gamma}{(ku_c')}, \quad l_\nu = d\left(\frac{\nu}{kd}\right)^{\frac{1}{3}}, \quad l_N = A^p d \tag{1.1}$$

is largest – the unsteady l_t, the viscous l_ν or the nonlinear l_N – the CL may be one of three types: unsteady, viscous or nonlinear. Here $u_c' = u'(y_c)$, d is the width of a shear flow, k and A are respectively the wavenumber and the dimensionless amplitude of the perturbation, $\gamma = |A^{-1}\,\mathrm{d}A/\mathrm{d}t|$, ν is the inverse of the Reynolds number, and $\frac{1}{2} \leqslant p \leqslant \frac{2}{3}$ depending on the Richardson number.

Formally, the difference in the properties of the CL is as follows. The eigenfunction ψ describing a neutrally stable solution for inviscid linearized equations, when $y = y_c$, exhibits a branching point, and the question immediately arises as to how to pass from the right-hand side of the CL ($y-y_c > 0$) to its left-hand side ($y-y_c < 0$). It appears that the general rule is thus: on the left-hand side of the CL $\psi(y-y_c) = \psi_A(|y-y_c|\,e^{i\phi})$, where ψ_A is an analytic continuation of the ψ defined for $y-y_c > 0$. In the case of an unsteady and viscous CL $\phi = -\pi$, i.e. 'indentation' is accomplished in a complex plane from below. This is Lin's rule (here and in what follows we assume $u_c' > 0$). For a nonlinear CL, $\phi = 0$ (Benney & Bergeron 1969; Davis 1969). This is one aspect of the so-called 'indentation' rule. Another aspect involves the problem of evaluating the diverging integrals appearing in the solvability conditions. As long as we use the linearized equations outside a CL, these aspects are both virtually indistinguishable: the integrals must be evaluated through indentation of the contour from below the singularity in the case of an unsteady or viscous CL, but in the sense of its principal value in the case of a nonlinear CL. Such is not the case, however, in nonlinear theory.

In order to avoid confusion in the terminology, we need to emphasize that one must distinguish between the linear or nonlinear type of CL and the linearity or nonlinearity of the problem. In speaking about the nonlinear problem we imply taking into account the nonlinear self-action (generation of harmonics and their back-reaction on the perturbation) which leads to the nonlinear evolution equation. In this case the nonlinearity may be the inner as well as the outer one (Ostrovsky, Stepanyants & Tsimring 1983). When the nonlinearity is the inner one the main contribution to the nonlinear interaction is due to the CL region, otherwise the role of the CL in the nonlinear interaction is not a dominant one, however, and contributions from all regions of the flow are of equal importance. Further, when we are treating the CL type, we imply the linear or nonlinear form of the equation inside the CL, or, in other words, the linear (viscous or unsteady) or nonlinear method of singularity regularization. Thus, one can consider the nonlinear CL in problems that are, in fact, linear. This approach has been used by, for example, Benney & Bergeron (1969) and Maslowe (1973) in their search for a new class of linear neutral modes for which the phase jump $\phi = 0$ (rather than $\phi = -\pi$ as in the 'old' linear theory). The problem concerning the outer nonlinearity and a viscous CL for a non-stratified flow was considered by Schade (1964), Benney & Maslowe (1975), Maslowe (1977*a*) and by Huerre (1977, 1980).

In problems regarding the nonlinear evolution, the CL regime plays the crucial role. However, as we shall demonstrate, there is no definite rule for evaluating the divergent integrals in this case and each particular problem requires careful analysis of the solution inside the CL.

We shall consider the nonlinear perturbation evolution in a weakly supercritical, stratified shear flow with a viscous CL (for such a flow, it appears more justified to refer to the CL as a dissipative one because the thermometric conductivity κ, together with viscosity ν, plays an equal role if the Prandtl number $\eta = \nu/\kappa \sim O(1)$; if, however, $\eta \gg 1$, the thermoconductive CL will be placed inside the viscous one). The case of a stratified flow has two advantages over the uniform case. First, the stable stratification when the supercriticality is weak, $\Delta J = \frac{1}{4}-J \ll 1$ (J is the Richardson number), leaves on the (k_x, k_z)-plane a small unstable region near $\boldsymbol{k}_0 = (k_0, 0)$ ($k_0 d \sim 1$, $|\boldsymbol{k}-\boldsymbol{k}_0|\,d \sim (\Delta J)^{\frac{1}{2}}$) and the increments of the perturbations are small. Here the two-dimensional formulation of the problem in the context of a weakly nonlinear theory is a natural one, in contrast to the uniform case including a wide range of unstable k, while investigation of a single, weakly unstable wave does not provide any insights into the fate of the system as a whole. Secondly, in a stratified flow, there

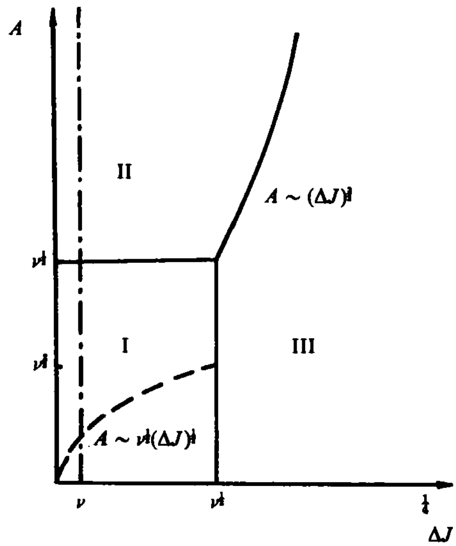

FIGURE 1. Diagram of different regimes of the critical layer. The dotted curve inside region I represents the saturation amplitude at Prandtl number $\eta < 1$.

exists a range of parameters in which the nonlinear evolution proceeds faster than the viscous spreading of the original flow so that this latter may be neglected (see also Brown, Rosen & Maslowe 1981; hereinafter referred to as Paper 1). In a uniform flow, however, in the viscous CL regime ($A \ll \nu^{2/3}$) the nonlinear time, in contrast, is very long and it is, therefore, appropriate to take into account the nonlinear effects only when an artificial force field keeping the original flow from viscous spreading is introduced. Huerre (1980) was the first to pay attention to this point.

Thus, a stratified flow permits a fully correct formulation of the nonlinear evolution problem in the viscous CL regime.

The linear theory of stratified flows with large Reynolds numbers ($\nu \equiv Re^{-1} \ll 1$) has been intensively developed since the beginning of the 1960s (Miles 1961; Drazin & Howard 1966), and a number of interesting results have recently been obtained (Ostrovsky *et al.* 1983; Makov & Stepanyants 1984). A stationary, nonlinear CL in such flows was considered by Kelley & Maslowe (1970) and Maslowe (1972, 1973).

Returning now to the nonlinear problem, we first outline the ranges of parameters where the various CL regimes take place. Since at weak supercriticality the linear increment $\gamma \sim \Delta J$ and the p in (1.1) is equal to $\frac{2}{3}$, then $l_t \sim \Delta J$, $l_\nu \sim \nu^{\frac{1}{3}}$, and $l_N \sim A^{\frac{2}{3}}$. Figure 1 is a diagram of the various CL regimes on the amplitude–supercriticality plane at a fixed Reynolds number. In region I, the CL is viscous; here $l_\nu \gg \max(l_t, l_N)$; and in region II, the CL is nonlinear: $l_N \gg \max(l_t, l_\nu)$; region III corresponds to an unsteady CL: $l_t \gg \max(l_\nu, l_N)$. In this work we shall restrict our attention to the nonlinear evolution of an initially small perturbation in the viscous CL regime, i.e. in region I of the parameters

$$A^2 \ll \nu \ll \Delta J \ll \nu^{\frac{1}{3}}.$$

Here the middle inequality excludes the domain $\Delta J \ll \nu$ where the viscous spreading of the original flow proceeds faster than the perturbation grows.

The analogous problem has been considered by a number of authors (Maslowe

1977*b*; Paper 1; Romanova & Tseitlin 1983), who have tried to obtain the nonlinear evolution equation. The most advanced of these is Brown *et al.*'s study (Paper 1)†.

In Paper 1 an estimate of the Landau constant is made according to which the most important contribution is of order $O(\nu^{-1})$ as $\nu \ll 1$, and by means of analytic methods it is shown that at Prandtl number $\eta = 1$ this main contribution vanishes. Also, by solving numerically the equations at finite Reynolds numbers $(20 < \nu^{-1} < 200)$ and $\eta \neq 1$ the fact that the Landau constant changes its sign at η not far from unity is verified and the Landau constant is found to be negative (stabilization) as $\eta < 1$ and positive (destabilization) as $\eta > 1$.

Unfortunately, the very much restricted character (only $\eta = 1$) of the analytic treatment together with the inconsistent use of the matched-asymptotic-expansions method (in Paper 1 only the particular fragments of the solution out of CL region are obtained and the inner solution matching to the outer one is not in fact made) provide the erroneous estimate $(O(\nu^{-\frac{2}{3}}))$ of the zeroth-harmonic (mean flow distortion) contribution to the Landau constant. Thus, Paper I evaluates the contribution of the second harmonic only. Our calculations demonstrate that the zeroth-harmonic contribution is also $O(\nu^{-1})$ and is approximately five times as large as the second-harmonic contribution. Fortunately, these contributions are both of the same sign and, therefore, the results of Paper 1 remain qualitatively valid. In addition, Paper 1 contains some more disputable statements, which we shall analyse in §6.

In this paper we have set ourselves several research and methodological objectives. First, we shall develop a consistent procedure for obtaining an analytic solution of the problem (or, more exactly, its uniformly valid expansion) outside the CL region (§3) as well as inside it (§4) at arbitrary Prandtl numbers by means of the matched-asymptotic-expansions method (e.g. Nayfeh 1981). Note that there are difficulties in the consistent evaluation of the zeroth harmonic, and in most papers mean flow distortion is ignored or is calculated incorrectly. In particular, in the problem under consideration, its calculation requires introducing an intermediate region situated between the CL and the outer region. The equations there have rather non-trivial solutions matching on the one side to an outer ('non steady') solution and on the other side to the inner ('dissipative') solution (§5).

Secondly, we shall provide an analysis of the asymptotic properties of the inner solution as a function of a complex variable. On this basis the question of an indentation rule of the singular point $y = y_c$ will be considered in detail (Appendix A).

Thirdly, we shall obtain a nonlinear evolution equation that takes correctly into account contributions to the Landau constant due to the zeroth harmonic as well as the second one (§§4 and 6).

Finally, we shall investigate carefully the solution at $\eta = 1$ when the main contribution to the Landau constant vanishes (Appendix B)‡.

The results are formulated and discussed in §6.

2. Formulation of the problem

Let us consider a shear flow in the gravity field $\boldsymbol{g}$. The unperturbed velocity in the x-direction depends on the vertical coordinate $y: u = U_0 \tanh(y/d)$. The scale of the density variation is assumed to be substantially larger than that of the velocity and

† We are grateful to Professor S. A. Maslowe for sending us a reprint of Paper 1 in response to a preprint of this paper of ours.

‡ Appendix B is available from the *Journal of Fluid Mechanics* Editorial Office.

we may regard the density ρ as a linear function of $y: \rho = \rho_0(0) + y\rho_0'(0)$. This is the Drazin (1958) model. Taking $|\rho_0' d|$, d and d/U_0 as the units of density, length and time respectively, the initial system of equations in the Boussinesq approximation can be written in the dimensionless form:

$$\frac{\partial}{\partial t}\Delta\psi - J\frac{\partial\rho}{\partial x} + \{\Delta\psi, \psi\} = \nu\Delta^2\psi, \tag{2.1}$$

$$\frac{\partial}{\partial t}\rho + \{\rho, \psi\} = \kappa\Delta\rho. \tag{2.2}$$

Here ψ is the stream function ($u_x = \partial\psi/\partial y$, $u_y = -\partial\psi/\partial x$), $\eta = \nu/\kappa$ is the Prandtl number, $\{a, b\} = (\partial a/\partial x)(\partial b/\partial y) - (\partial a/\partial y)(\partial b/\partial x)$, and $J = d^2|\rho_0'(0)|g/U_0^2\rho_0(0)$ is the Richardson number. In the inviscid limit, the neutral curve for the model was obtained by Drazin (1958): on the (J, k)-plane it is determined by the equation $J = Ri(k)$, $Ri = k^2(1-k^2)$, where k is the longitudinal wavenumber of the perturbation.

The purpose of the subsequent calculations is to derive the equation for the wave having the maximal increment in the linear theory, i.e. the equation in the form (Landau & Lifshitz 1953):

$$\frac{\partial A}{\partial t} = \gamma_{\max} A + a|A|^2 A. \tag{2.3}$$

Let us introduce the small parameter ϵ characterizing the perturbation amplitude at which the linear and nonlinear terms on the right-hand side of (2.3) are of the same order of magnitude (if $\eta < 1$, it is the saturation amplitude). As will be shown, the Landau constant $a = O(\kappa^{-1})$, hence $\gamma_{\max}$, $\Delta J = O(\epsilon^2/\kappa)$. The range of parameters of interest is defined by the inequalities

$$\epsilon^2 \ll \kappa \ll \mu \ll \kappa^{\frac{1}{3}}, \quad \mu = \frac{\epsilon^2}{\kappa}. \tag{2.4}$$

Guided by this ordering, we introduce the stretched time $\tau = \mu t$ and represent the Richardson number as

$$J = \tfrac{1}{4} + \mu J^{(1)}. \tag{2.5}$$

Note that for κ, $\nu \neq 0$, the neutral curve is displaced (Maslowe & Thompson 1971) but this displacement is negligibly small compared with the effects taken into account in (2.3). After extracting the perturbed parts of ψ and ρ, we expand them into a Fourier series

$$\psi = \ln(\cosh y) + \sum_{l=-\infty}^{\infty} \psi_l(\tau, y)\, e^{ikl x}, \quad \rho = \rho_0(0) - y + \sum_{l=-\infty}^{\infty} \rho_l(\tau, y)\, e^{iklx}, \tag{2.6}$$

where $\psi_{-l} = \bar{\psi}_l$, $\rho_{-l} = \bar{\rho}_l$, and the overbar denotes the complex conjugate. The ψ_l and ρ_l functions depend on the small parameters ϵ, μ and κ (only two of these are independent) and in calculating them we shall develop appropriate expansions. In order to derive the evolution equation (2.3), we need the fundamental ($l = \pm 1$), the second ($l = \pm 2$) and the zeroth ($l = 0$) harmonics.

Equations (2.1) and (2.2) involve the small parameter multiplying the higher derivative. The intrinsic region of the ψ_l and ρ_l fast variations – the critical layer – is localized in the Drazin model (and in each flow with an odd velocity profile) at $y = 0$. Depending on the relative role of unsteadiness ($(\partial/\partial t)\Delta\psi \sim \mu l^{-2}\psi$) and dissipative effects ($\nu\Delta^2\psi \sim \kappa l^{-4}\psi$), the uniformly valid approximation of the solution of (2.1) and (2.2) is constructed in a different way in various regions of the flow. Initially, we shall solve the outer problem, i.e. we calculate ψ_l and ρ_l in the $|y| \gg (\kappa/\mu)^{\frac{1}{3}} = \kappa/\epsilon$ region where the non-stationarity dominates. Here the obvious boundary conditions are

$$\psi_l \to 0, \quad \rho_l \to 0 \quad \text{as } y \to \pm\infty. \tag{2.7}$$

The functions ψ_l and ρ_l, obtained at $y < 0$ and $y > 0$, are to be matched using the solutions into the CL ($|y| \lesssim \kappa^{\frac{1}{3}}$) where dissipative effects dominate and in the intermediate region where the dissipative and unsteady effects are of equal importance. Note that the intermediate region plays an important role for the zeroth harmonic only, while for the other harmonics it appears to be a trivial transmission link between the outer region and the CL.

3. The outer problem

Let us consider a wave of sufficiently small amplitude in the neighbourhood of the neutral-curve maximum ($J = \frac{1}{4}$, $k = k_0 = 1/\surd 2$). The fundamental harmonic dominates in the perturbation and has an amplitude of order ϵ. The second and the zeroth harmonics are the result of the self-influence and have an amplitude of order ϵ^2.

3.1. *The fundamental harmonic*

In order to derive the evolution equation, not only the main term but also the following terms of the expansion

$$\psi_1 = \epsilon\psi_1^{(1)} + \epsilon\mu\psi_1^{(2)} + \epsilon\kappa\psi_1^{(3)} + \dots \tag{3.1}$$

are needed. Here $\psi_1^{(1)}$ is the neutral mode of an inviscid linear problem, $\psi_1^{(2)}$ takes into account the τ-dependence of the solution and is also necessary for the zeroth-harmonic evaluation in the outer region. And $\psi_1^{(3)}$ is a correction to $\psi_1^{(1)}$ for dissipative effects and is used for the zeroth-harmonic calculation in the intermediate region.

The function $\psi_1^{(1)}$ is a solution of the Taylor–Goldstein equation which, in our problem, has the form

$$\mathrm{L}_1 \psi_1^{(1)} = 0, \tag{3.2}$$

where the operator L_l is

$$\mathrm{L}_l = \frac{\partial^2}{\partial y^2} - \left[(lk_0)^2 - \frac{2}{\cosh^2 y} - \tfrac{1}{4}\tanh^{-2} y\right]. \tag{3.3}$$

A solution of (3.2) with the boundary conditions (2.7) is

$$\psi_1^{(1)} = A^{\pm}(\tau)\,\phi_a(y), \tag{3.4}$$

where $\phi_a = \sinh^{\frac{1}{2}}|y|\,\mathrm{sech}\, y$ ($\pm$ are related to $y \gtrless 0$ respectively); a relationship between A^+ and A^- will be given by matching to the inner problem (from linear theory it is clear that $A^- = -\mathrm{i}A^+$ ought to be obtained).

The following terms of expansion (3.1) obey equations of the form

$$\mathrm{L}_1 \psi_1^{(m)} = Q^{(m)}, \quad m = 2, 3, \dots, \tag{3.5}$$

which have the solutions

$$\psi_1^{(m)} = \int^y \mathrm{d}z[\phi_a(y)\,\phi_b(z) - \phi_a(z)\,\phi_b(y)]\,Q^{(m)}(z) + (a_1^{(m)\pm}\phi_a + b_1^{(m)\pm}\phi_b)\,A^{\pm},$$

where ϕ_b is the second solution of (3.2) (that is a linearly independent one with ϕ_a): $\phi_b = -\phi_a(y)\int^y \mathrm{d}z\,\phi_a^{-2}(z)$, i.e.

$$\phi_b = s\phi_a[1 - \cosh y - \tfrac{1}{2}\Lambda(y)], \quad \Lambda = \ln|\tanh^2 \tfrac{1}{2}y|, \quad s = \mathrm{sgn}\,(y). \tag{3.6}$$

Note that ϕ_b increases as $\exp(\frac{1}{2}|y|)$ when $y \to \pm\infty$. Introducing

$$F^{(m)}(y) = \int^y \mathrm{d}z\, \phi_a(z)\, Q^{(m)}(z)$$

we can represent $\psi_1^{(m)}$ as

$$\psi_1^{(m)} = \phi_a(y) \int^y \mathrm{d}z\, [F^{(m)}(z) - b_1^{(m)\pm} A^{\pm}]\, \phi_a^{-2}(z) + a_1^{(m)\pm} A^{\pm} \phi_a(y). \tag{3.7}$$

Here we encounter the so-called modified solvability condition (MSC) (the term introduced by Benney & Maslowe 1975). If the operator L_1 were non-singular at $y = 0$, the necessary and sufficient condition for solvability of (3.5) would be the orthogonality of $Q^{(m)}$ to the eigenfunction of the homogeneous equation:

$$\int_{-\infty}^{\infty} Q^{(m)}(z)\, \phi_a(z)\, \mathrm{d}z = 0.$$

When $y = 0$, however, this integral diverges owing to a singularity of L_1. The MSC is obtained from the boundary conditions (2.7) which require convergence of the integral in (3.7) when $y \to \pm\infty$. This yields

$$F^{(m)}(\pm\infty) = b_1^{(m)\pm} A^{\pm}. \tag{3.8}$$

The unknown coefficients will be determined from the inner problem.

For $m = 2, 3$ we obtain ($u = \tanh y$)

$$Q^{(2)} = \frac{\mathrm{i}}{k_0 u} \frac{\partial A^{\pm}}{\partial \tau} \left(\Delta - \frac{\frac{1}{4}}{u^2}\right) \phi_a - \frac{J^{(1)}}{u^2} A^{\pm} \phi_a; \quad Q^{(3)} = -\frac{\mathrm{i}}{k_0} A^{\pm} \left(\frac{\eta}{u} \Delta^2 \phi_a - \frac{\frac{1}{4}}{u^2} \Delta \frac{\phi_a}{u}\right),$$

$$F^{(2)}(y) = -\frac{\mathrm{i}}{k_0} \frac{\partial A^{\pm}}{\partial \tau} \left(\frac{u}{\cosh y} - \frac{\frac{1}{2}}{\sinh y} + \arctan(\sinh y)\right) s - \tfrac{1}{2} J^{(1)} A^{\pm} s A(y),$$

$$F^{(3)}(y) = -\frac{\mathrm{i}}{k_0} \left[\eta \left(\frac{6u}{\cosh^3 y} + \frac{3u}{\cosh y} + \frac{\frac{5}{16}}{\sinh^3 y} - \frac{\frac{9}{16}}{\sinh y} + \tfrac{5}{2} \arctan(\sinh y)\right) + \frac{1}{16} \left(\frac{1}{\sinh^3 y} - \frac{1}{\sinh y}\right)\right] A^{\pm} s,$$

and, according to (3.8),

$$b_1^{(2)\pm} = -\frac{\mathrm{i}\pi}{2k_0} \frac{\partial A^{\pm}}{\partial \tau} \frac{1}{A^{\pm}}, \quad b_1^{(3)\pm} = -\frac{5\mathrm{i}\pi}{4k_0} \eta.$$

Deriving the evolution equation will require the first ($m = 2$) MSC. It can be written as

$$b_1^{(2)+} + b_1^{(2)-} = -\frac{\mathrm{i}\pi}{k_0} \frac{1}{A^+} \frac{\partial A^+}{\partial \tau} = -\frac{\mathrm{i}\pi}{k_0} \frac{1}{A^-} \frac{\partial A^-}{\partial \tau}. \tag{3.9}$$

From the inner problem, we shall find that $b_1^{(2)+} + b_1^{(2)-} = \mathrm{i}\pi (J^{(1)} - (a/k_0)\, |A|^2)$ and shall calculate the Landau constant a. Thus, it is the MSC (3.9) that yields the evolution equation (2.3).

The second ($m = 3$) MSC provides the correction for viscosity to the critical Richardson number: $J_0 = \frac{1}{4} - 5\nu/4k_0$ (cf. Maslowe & Thompson 1971). It is interesting to note that in the Drazin model this correction does not depend on κ. As we have mentioned above, its evaluation is not necessary in our problem.

Let us display the function $\psi_1^{(2)}$, $\rho_1^{(2)}$ and $\psi_1^{(3)}$, $\rho_1^{(3)}$:

$$\psi_1^{(2)} = -\frac{\mathrm{i}}{k_0}\frac{\partial A^\pm}{\partial \tau}\left[g(y)+\tfrac{1}{2}y+\frac{\tfrac{1}{2}}{u}\right]\phi_a - \tfrac{1}{2}J^{(1)}A^\pm[\Lambda(y)\cosh y$$

$$-2\ln|\sinh y| + \tfrac{1}{4}\Lambda^2(y)]\,\phi_a + (a_1^{(2)\pm}\phi_a + b_1^{(2)\pm}\phi_b)\,A^\pm,$$

$$\rho_1^{(2)} = -\frac{\psi_1^{(2)}}{u} - \frac{\mathrm{i}}{k_0}\frac{\partial A^\pm}{\partial \tau}\frac{\phi_a}{u^2},$$

$$\psi_1^{(3)} = -\frac{\mathrm{i}}{k_0}A^\pm\left[\eta\left(\tfrac{5}{2}g(y) - \frac{\tfrac{5}{48}}{u^3} + \frac{\tfrac{9}{16}}{u} + 6u + \tfrac{39}{16}y\right) - \frac{1}{16}\left(\frac{\tfrac{1}{3}}{u^3} - \frac{1}{u} + y\right)\right]\phi_a$$

$$+ (a_1^{(3)\pm}\phi_a + b_1^{(3)\pm}\phi_b)\,A^\pm;$$

$$\rho_1^{(3)} = -\frac{\psi_1^{(3)}}{u} + \frac{\mathrm{i}}{4k_0}A^\pm\left(\frac{3}{\sinh^4 y} + \frac{2}{\sinh^2 y} - 1\right)\phi_a,$$

where

$$g(y) = \int_0^y \frac{\arctan(\sinh z)}{\sinh z}\cosh^2 z\,\mathrm{d}z.$$

3.2. *The second harmonic*

We only need the main term of the expansion $\psi_2 = \epsilon^2\psi_2^{(1)} + \dots$, which obeys the equation

$$\mathrm{L}_2\,\psi_2^{(1)} = \left(2\frac{\sinh y}{\cosh^4 y} + \frac{\tfrac{3}{8}}{\sinh^3 y}\right)(A^\pm)^2 s. \tag{3.10}$$

For matching to the inner solution, only the asymptotic expansion of $\psi_2^{(1)}$ as $y \to 0$ is needed. The main term is easily calculated (see (3.12)) and a more detailed analysis of this expansion necessary for the $\eta = 1$ case is made in Appendix B.

3.3. *The zeroth harmonic*

Here we also need only the main term of the expansion $\psi_0 = \epsilon^2\psi_0^{(1)} + \dots$ but it can be determined in the $\epsilon^2\mu$-order only because in ϵ^2 the equations for the zeroth harmonic reduce to identities $(0 = 0)$. We obtain

$$\frac{\partial}{\partial \tau}\frac{\partial^2 \psi_0^{(1)}}{\partial y^2} = -\frac{\partial}{\partial y}\left[\frac{\partial |A^\pm|^2}{\partial \tau}\left(\frac{\cosh y}{2\sinh^2 y} + \frac{2}{\cosh^3 y}\right)s\right]; \quad \frac{\partial \rho_0^{(1)}}{\partial \tau} = -\frac{\partial}{\partial y}\left(\frac{\partial |A^\pm|^2}{\partial \tau}\frac{s}{\sinh y}\right),$$

whence, by assuming adiabatic switching on, the mean velocity and density perturbations can be expressed as

$$\frac{\partial \psi_0^{(1)}}{\partial y} = -\left(\frac{\cosh y}{2\sinh^2 y} + \frac{2}{\cosh^3 y}\right)|A^\pm|^2 s; \quad \rho_0^{(1)} = \frac{\cosh y}{\sinh^2 y}|A^\pm|^2 s.$$

Thus, our construction of the outer solution is completed. For matching, we need the asymptotic expansion of an outer solution at $|y| \ll 1$. Since the intermediate region is only essential for the zeroth harmonic, the expansions of the fundamental and the second harmonic will be expressed in terms of an inner coordinate $Y = \kappa^{-\frac{1}{3}}y$. We obtain:

the fundamental

$$\psi_1 = \epsilon\kappa^{\frac{1}{6}}A^\pm|Y|^{\frac{1}{2}}\left[1 + \frac{\mathrm{i}(5\eta+1)}{48k_0}Y^{-3}\right] - \epsilon^3\kappa^{-\frac{7}{6}}\frac{\mathrm{i}}{2k_0}\frac{\partial A^\pm}{\partial \tau}\frac{|Y|^{\frac{1}{2}}}{Y}$$

$$+ \epsilon^3\kappa^{-\frac{5}{6}}A^\pm|Y|^{\frac{1}{2}}[\tfrac{1}{2}J^{(1)}(2\,ln\,2 - \lambda^2) + a_1^{(2)\pm} - b_1^{(2)\pm}\lambda\cdot s], \tag{3.11}$$

$$\lambda = \ln(\tfrac{1}{2}\kappa^{\frac{1}{3}}|Y|);$$

the second harmonic

$$\psi_2 = \epsilon^2\kappa^{-\frac{1}{3}}\frac{(A^{\pm})^2}{6Y}s+\dots; \tag{3.12}$$

the zeroth harmonic

$$\psi_0 = \epsilon^2\frac{|A^{\pm}|^2}{2y}s+\dots,\quad \rho_0 = \epsilon^2\frac{|A^{\pm}|^2}{y^2}s+\dots. \tag{3.13}$$

4. The inner solution

For the inner problem, it is convenient to write the original set of equations in the form

$$\left(\eta\mathscr{D}^4 - Y\frac{\partial}{\partial x}\mathscr{D}^2\right)\Psi+\frac{1}{4}\frac{\partial P}{\partial x} = \mathscr{R}_1 \equiv \kappa^{-\frac{2}{3}}\{\Psi_{YY},\Psi\}^*+\epsilon^2\kappa^{-\frac{4}{3}}\frac{\partial}{\partial \tau}\Psi_{YY}-\epsilon^2\kappa^{-1}J^{(1)}P_x+\dots, \tag{4.1}$$

$$\left(\mathscr{D}^2 - Y\frac{\partial}{\partial x}\right)P-\frac{\partial}{\partial x}\Psi = \mathscr{R}_2 \equiv \kappa^{-\frac{2}{3}}\{P,\Psi\}^*+\epsilon^2\kappa^{-\frac{4}{3}}\frac{\partial P}{\partial \tau}+\dots. \tag{4.2}$$

Here ... stands for omitted terms not needed for the subsequent calculations. The ψ and ρ are replaced here by Ψ and P:

$$\psi = \tfrac{1}{2}\kappa^{\frac{2}{3}}Y^2+\Psi,\quad \rho = \rho_0(0)-\kappa^{\frac{1}{3}}Y+\kappa^{-\frac{1}{3}}P,$$

and the following notation is introduced: $\{a, b\}^* = a_x b_Y - a_Y b_x$, $\mathscr{D} = \partial/\partial Y$. Both the inner and the outer solutions are represented as the sum of the harmonics

$$\Psi = \sum_{l=-\infty}^{\infty}\Psi_l(\tau, Y)\,\mathrm{e}^{\mathrm{i}lk_0x},\quad P = \sum_{l=-\infty}^{\infty}P_l(\tau, Y)\,\mathrm{e}^{\mathrm{i}lk_0x},$$

and for each harmonic, expansion will proceed in terms of ϵ and κ. Comparison with (3.11)–(3.13) indicates that the following terms are needed:

the fundamental

$$\Psi_1 = \epsilon\kappa^{\frac{1}{3}}\Psi_1^{(1)}+\epsilon^3\kappa^{-\frac{7}{3}}\Psi_1^{(2)}+\epsilon^3\kappa^{-\frac{5}{3}}\Psi_1^{(3)}+\dots;$$

the second harmonic

$$\Psi_2 = \epsilon^2\kappa^{-\frac{1}{3}}\Psi_2^{(1)}+\dots;$$

the zeroth harmonic

$$\Psi_0 = \epsilon^2\kappa^{-\frac{1}{3}}\Psi_0^{(1)}+\dots.$$

As before, the superscript indicates the consecutive number of iterations.

At each of these orders, it is necessary to obtain the general solution of (4.1) and (4.2) for each harmonic, to calculate its asymptotic expansion and to match it to one of the outer solutions (3.11) and (3.12). The pertinent analysis of the asymptotic properties of a compound (outer + inner) solution is made in Appendix A. As the last step, at $O(\epsilon^3\kappa^{-\frac{5}{3}})$ of the fundamental the sum $b_1^{(2)+}+b_1^{(2)-}$ is obtained and is then used to derive the evolution equation from (3.9).

4.1. $O(\epsilon\kappa^{\frac{1}{3}})$ *of the fundamental*

$$\mathscr{L}_1\Psi_1^{(1)} = 0,\quad P_1^{(1)} = \frac{4\mathrm{i}}{k_0}(\eta\mathscr{D}^2-\mathrm{i}k_0Y)\mathscr{D}^2\Psi_1^{(1)}, \tag{4.3}$$

where

$$\mathscr{L}_n = \eta\mathscr{D}^6-\mathrm{i}nk_0(\eta Y\mathscr{D}^4+\mathscr{D}^2Y\mathscr{D}^2)-(nk_0)^2(Y^2\mathscr{D}^2+\tfrac{1}{4}). \tag{4.4}$$

The sixth-order equation (with J arbitrary instead of $\frac{1}{4}$ but $\eta = 1$) was investigated by Koppel (1964) using the Laplace transform. We shall need the following result (for η arbitrary), which can be obtained from the asymptotic properties of (4.3) in the complex-Y plane (see Appendix A): in the $-\frac{7}{6}\pi < \arg Y < \frac{1}{6}\pi$ sector ($\mathscr{M}$-region) two of the six linearly independent solutions of (4.3), W_a and W_b, have as $|Y| \to \infty$ the asymptotic representations

$$\left.\begin{aligned} W_a(Y) &= Y^{\frac{1}{2}}(1+O(Y^{-3})),\\ W_b(Y) &= Y^{\frac{1}{2}} \ln(\tfrac{1}{2}\kappa^{\frac{1}{3}} Y)(1+O(Y^{-3})),\end{aligned}\right\} \tag{4.5}$$

and the remaining four solutions exponentially grow either as $Y \to +\infty$ or as $Y \to -\infty$ (we have specially introduced $\kappa^{\frac{1}{3}}$ in W_b to simplify the matching to the outer solution). Thus we have

$$\Psi_1^{(1)} = A W_a(Y), \quad P_1^{(1)} = \frac{4\mathrm{i}}{k_0} A(\eta \mathscr{D}^4 - \mathrm{i}k_0 Y \mathscr{D}^2) W_a \equiv A\tilde{P}_1. \tag{4.6}$$

The matching to (3.11) provides $A^+ = A$; $A^- = -\mathrm{i}A$. Note that the $\mathscr{L}_n$ operators are invariant to the replacement Y by $-Y$ with simultaneous complex conjugation. Owing to this, $\overline{W}_a(-Y) = \mathrm{i}W_a(Y)$.

4.2. $O(\epsilon^2\kappa^{-\frac{1}{3}})$ *of the second harmonic*

$$\mathscr{L}_2 \Psi_2^{(1)} = \mathscr{R} \equiv (\mathscr{D}^2 - 2\mathrm{i}k_0 Y)\mathscr{R}_1 - \tfrac{1}{2}\mathrm{i}k_0 \mathscr{R}_2, \tag{4.7}$$

$$\mathscr{R}_1 = \mathrm{i}k_0 A^2(W_a'^2 - W_a'' W_a)', \quad \mathscr{R}_2 = \mathrm{i}k_0 A^2(\tilde{P}_1 W_a' - W_a \tilde{P}_1').$$

The prime denotes the Y-derivative. As the right-hand side of (4.7) is analytic in $\mathscr{M}$ (see Appendix A), the $\Psi_2^{(1)}$ asymptotic expansion as $Y \to \pm\infty$ is readily determined from those of $\mathscr{R}$ and is automatically matched to (3.12). We can write

$$\left.\begin{aligned} \Psi_2^{(1)} &= A^2\Phi_2,\\ P_2^{(1)} &= A^2\left\{\frac{2\mathrm{i}}{k_0}(\eta\mathscr{D}^2 - 2\mathrm{i}k_0 Y)\mathscr{D}^2\Phi_2 + 2\mathscr{D}(W_a'^2 - W_a'' W_a)\right\} \equiv A^2\tilde{P}_2.\end{aligned}\right\} \tag{4.8}$$

Note that

$$\overline{\Phi}_2(-Y) = -\Phi_2(Y), \quad \overline{\tilde{P}}_2(-Y) = \tilde{P}_2(Y)$$

4.3. $O(\epsilon^2\kappa^{-\frac{1}{3}})$ *of the zeroth harmonic*

$$\left.\begin{aligned} \eta\mathscr{D}^4\Psi_0^{(1)} &= \mathrm{i}k_0|A|^2\mathscr{D}^2(W_a'\overline{W}_a - \text{c.c.}),\\ \mathscr{D}^2 P_0^{(1)} &= \mathrm{i}k_0|A|^2\mathscr{D}(\tilde{P}_1\overline{W}_a - \text{c.c.}).\end{aligned}\right\} \tag{4.9}$$

Integrating (4.9) we obtain for mean velocity and density perturbations

$$\left.\begin{aligned} \Psi_0^{(1)\prime} &= \frac{\mathrm{i}k_0}{\eta}|A|^2\int_0^Y \mathrm{d}z\,[W_a'(z)\,\overline{W}_a(z) - \text{c.c.}] \equiv |A|^2\Phi_0(Y),\\ P_0^{(1)\prime} &= 4|A|^2[\mathrm{i}k_0 Y(W_a''\overline{W}_a - \text{c.c.}) - \eta(W_a^{\mathrm{IV}}\overline{W}_a + \text{c.c.})] \equiv |A|^2\tilde{P}_0.\end{aligned}\right\} \tag{4.10}$$

Note that $\Psi_0^{(1)}$ and $P_0^{(1)}$ are analytic in the $|\arg Y| < \frac{1}{6}\pi$ and $-\frac{7}{6}\pi < \arg Y < -\frac{5}{6}\pi$ sectors only. The mean velocity perturbation is an odd real function of Y and has the jump through the CL

$$\Psi_0^{(1)\prime}(+\infty) - \Psi_0^{(1)\prime}(-\infty) = 2\Phi_0(\infty)|A|^2. \tag{4.11}$$

The asymptotic expansions as $Y \to \pm\infty$ are

$$\Psi_0^{(1)} = \frac{|A|^2}{16Y}\left(5 + \frac{1}{\eta}\right)s + |A|^2\Phi_0(\infty)|Y| + \ldots, \quad P_0^{(1)} = \frac{3|A|^2}{4Y^2}s + \ldots, \tag{4.12}$$

and they do not match to (3.13) immediately. Matching requires evaluating the zeroth harmonic in the intermediate region $\kappa^{\frac{1}{3}} \ll y \ll \kappa/\epsilon$, see §5.

4.4. $O(\epsilon^3\kappa^{-\frac{7}{6}})$ *of the fundamental*

$$\mathscr{L}_1 \Psi_1^{(2)} = [(\eta+1)\mathscr{D}^4 - 2\mathrm{i}k_0\, Y\mathscr{D}^2]\, W_a \frac{\partial A}{\partial \tau}. \tag{4.13}$$

By analogy with (4.7), the solution is automatically matched to (3.11).

4.5. $O(\epsilon^3\kappa^{-\frac{5}{6}})$ *of the fundamental*

$$\mathscr{L}_1 \Psi_1^{(3)} = \mathscr{R}, \tag{4.14}$$

$$\mathscr{R} = (\mathscr{D}^2 - \mathrm{i}k_0\, Y)\mathscr{R}_1 - \tfrac{1}{4}\mathrm{i}k_0 \mathscr{R}_2 \equiv \mathscr{R}_N |A|^2 A + k_0^2 J^{(1)} A W_a;$$

$$\mathscr{R}_1 = |A|^2 A \mathscr{D}\,[\{\Phi_2', \overline{W}_a\}^* + \{\overline{W}_a', \Phi_2\}^* + \mathrm{i}k_0(W_a'\Phi_0 - W_a\Phi_0')] - \mathrm{i}k_0 J^{(1)} P_1^{(1)}$$

$$\equiv |A|^2 A \mathscr{R}_{1N} - \mathrm{i}k_0 J^{(1)} A \tilde{P}_1,$$

$$\mathscr{R}_2 = |A|^2 A\,[\{\tilde{P}_2, \overline{W}_a\}^* + \{\overline{\tilde{P}}_1, \Phi_2\}^* + \mathrm{i}k_0(\tilde{P}_1\Phi_0 - \tilde{P}_0 W_a)] \equiv |A|^2 A \mathscr{R}_{2N}.$$

The right-hand side of (4.14) (or more exactly, the term $|A|^2 A\, \mathscr{R}_N$) is non-analytic in the lower half-plane (in the $-\frac{5}{6}\pi < \arg Y < -\frac{1}{6}\pi$ sector); therefore $\Psi_1^{(3)}$ is also non-analytic in $\mathscr{M}$ and, although the asymptotic representation of $\mathscr{R}$ as $Y \to \pm\infty$,

$$\mathscr{R} \sim -\tfrac{1}{4}|A|^2 A \Phi_0(\infty)\, Y^{-\frac{1}{2}s} + k_0^2 J^{(1)} A\, Y^{\frac{1}{2}} + O(Y^{-\frac{5}{2}}),$$

allows us to determine the general form of the $\Psi_1^{(3)}$ asymptotic expansion

$$\Psi_1^{(3)} = -\tfrac{1}{2} J^{(1)} A\, Y^{\frac{1}{2}} \ln^2(\tfrac{1}{2}\kappa^{\frac{1}{3}} Y) + \tfrac{1}{2}\Phi_0(\infty)\, |A|^2 A\, Y^{-\frac{1}{2}s}$$
$$+ [m^{\pm} + n^{\pm} \ln(\tfrac{1}{2}\kappa^{\frac{1}{3}}\, Y)]\, A\, Y^{\frac{1}{2}} + O(Y^{-\frac{5}{2}}), \tag{4.15}$$

the solution of (4.14) along the real axis is required for evaluating $m^+ - m^-$ and $n^+ - n^-$. We emphasize that these quantities are non-zero in general (see Appendix A).

Matching (4.15) to (3.11) gives $b_1^{(2)+} = -n^+$, $b_1^{(2)-} = n^- + \mathrm{i}\pi J^{(1)}$, whence

$$b_1^{(2)+} + b_1^{(2)-} = -(n^+ - n^-) + \mathrm{i}\pi J^{(1)}. \tag{4.16}$$

In order to calculate $(n^+ - n^-)$ we multiply (4.14) by the eigenfunction v of the equation conjugate to (4.3):

$$\mathscr{L}_1^+ v \equiv [\eta\mathscr{D}^6 - \mathrm{i}k_0(\eta\mathscr{D}^4 + \mathscr{D}^2 Y \mathscr{D}^2) - k_0^2(\mathscr{D}^2 Y^2 + \tfrac{1}{4})]\, v = 0, \tag{4.17}$$

$$v \sim Y^{-\frac{3}{2}} \quad \text{when } Y \in \mathscr{M},$$

and integrate it for Y between $-\infty$ and ∞:

$$\int_{-\infty}^{\infty} \mathrm{d}Y\, v \mathscr{L}_1 \Psi_1^{(3)} = \int_{-\infty}^{\infty} \mathrm{d}Y\, \mathscr{R} v.$$

On integrating by parts on the left-hand side and using (4.15) and (4.17), we obtain

$$A(n^+ - n^-) = -k_0^{-2} \int_{-\infty}^{\infty} \mathrm{d}Y\, \mathscr{R} v = -k_0^{-2} |A|^2 A \int_{-\infty}^{\infty} \mathrm{d}Y\, \mathscr{R}_N v. \tag{4.18}$$

We have retained in $\mathscr{R}$ only the 'nonlinear term' since the analytic in $\mathscr{M}$ term does not contribute to $n^+ - n^-$ (see Appendix A). Using the transformation properties of W_a, Φ_2, Φ_0 and (4.17) it can be easily seen that $\overline{\mathscr{R}_N}(-Y)\,\bar{v}(-Y) = -\mathscr{R}_N(Y)\, v(Y)$, i.e.

the real part of the integrand is odd and the imaginary part is even. Substitution of (4.16) and (4.18) into the MSC (3.9) provides the evolution equation of the form (2.3)

$$\frac{\partial A}{\partial \tau} = -J^{(1)}k_0 A - \frac{2}{\pi k_0}|A|^2 A \,\mathrm{Im} \int_0^\infty \mathscr{R}_N v \,\mathrm{d}Y. \tag{4.19}$$

The Landau constant (or more exactly, the main contribution to it)

$$a = -\frac{2}{\pi k_0}\int_0^\infty \mathrm{d}Y \,\mathrm{Im}\,(\mathscr{R}_N v) \tag{4.20}$$

depends on the Prandtl number and is equal to zero at $\eta = 1$. Indeed, in this case, the $\mathscr{L}_n$ operators are factorized,

$$\mathscr{L}_n = \mathscr{N}_n^2, \quad \mathscr{N}_n = \mathscr{D}^3 - \mathrm{i}nk_0 Y\mathscr{D} + \tfrac{1}{2}\mathrm{i}nk_0,$$

W_a is the solution of the equation $\mathscr{N}_1 W_a = 0$ and the conjugate problem $\mathscr{N}_1^+ v = 0$ has the solution $v = -4W_a''$. It is easy to see that for $O(\epsilon\kappa^{\frac{1}{3}})$ of the fundamental and for $O(\epsilon^2\kappa^{-\frac{1}{3}})$ of the second and zeroth harmonics, the relations $P_n^{(m)} = -2\mathscr{D}\Psi_n^{(m)}$ is valid. Turning to the right-hand side of (4.14) we can see that $\mathscr{R}_{1N} = -\frac{1}{2}\mathscr{D}\mathscr{R}_{2N}$ and $\mathscr{R}_N = -\frac{1}{2}\mathscr{N}_1\mathscr{R}_{2N}$, wherefrom we obtain

$$a = \frac{\mathrm{i}}{\pi k_0}\int_{-\infty}^{\infty} \mathscr{R}_N v \,\mathrm{d}Y = -\frac{\mathrm{i}}{2\pi k_0}\int_{-\infty}^{\infty} \mathrm{d}Y v \mathscr{N}_1 \mathscr{R}_{2N} = -\frac{\mathrm{i}}{2\pi k_0}\int_{-\infty}^{\infty} \mathrm{d}Y \mathscr{R}_{2N}\mathscr{N}_1^+ v = 0.$$

This result is only valid in the limit of high Reynolds numbers ($\nu \rightarrow 0$). In fact, the Landau constant is equal to zero at $\eta = \eta_0$, where

$$\eta_0 = 1 + \kappa^{\frac{1}{3}}h_1 + \kappa^{\frac{2}{3}}h_2 + \kappa h_3 + \dots. \tag{4.21}$$

In other words, at $\eta = 1$ the Landau constant has an expansion

$$a = a_1\kappa^{-\frac{2}{3}} + a_2\kappa^{-\frac{1}{3}} + a_3 + \dots \tag{4.22}$$

and is $O(\kappa^{-\frac{2}{3}})$ rather than $O(\kappa^{-1})$ as in the general case. The numerical values of h_i and a_i depend on the original flow model. In particular, for the Drazin model $h_1 = 0.47$, $a_1 = -0.054$ while in the Holmboe model (see Gossard & Hooke 1975), $h_1 = 2.28$, $a_1 = -0.26$. In more details the case of Prandtl number being unity is considered in Appendix B.

It is convenient to separate the contributions to the Landau constant a due to the second and zeroth harmonics (see (4.14))

$$a = a^{(2)} + a^{(0)}. \tag{4.23}$$

Note that at $\eta = 1$ both contributions to (4.23) go to zero.

Evaluating the Landau constant at $\eta \neq 1$ requires a numerical procedure. The integration of (4.3), (4.7), (4.10) and (4.17) which determine the fundamental, the second and the zeroth harmonics and a solution of the conjugate problem respectively is accomplished at $0 < Y < \infty$ (or more exactly, at $0 < Y < Y_{max}$, $Y_{max} \gg 1$) by the Runge–Kutta procedure. The non-triviality of the numerical procedure (let us follow it for the example of a W_a calculation) is due to the fact that the asymptotic values of the function and its five derivatives are set at $Y = Y_{max}$ as boundary conditions. In such a formulation of the Cauchy problem, the eigenfunctions of the homogeneous equation that are exponentially small as $Y \rightarrow +\infty$ (see Appendix A) will enter into the solution with uncontrollable weight and this will substantially distort the solution at finite Y. Therefore, we have used the fact that on the Stokes line $\arg Y = -\frac{1}{2}\pi$ the exponential terms mentioned above are comparable with (4.5) and they can be

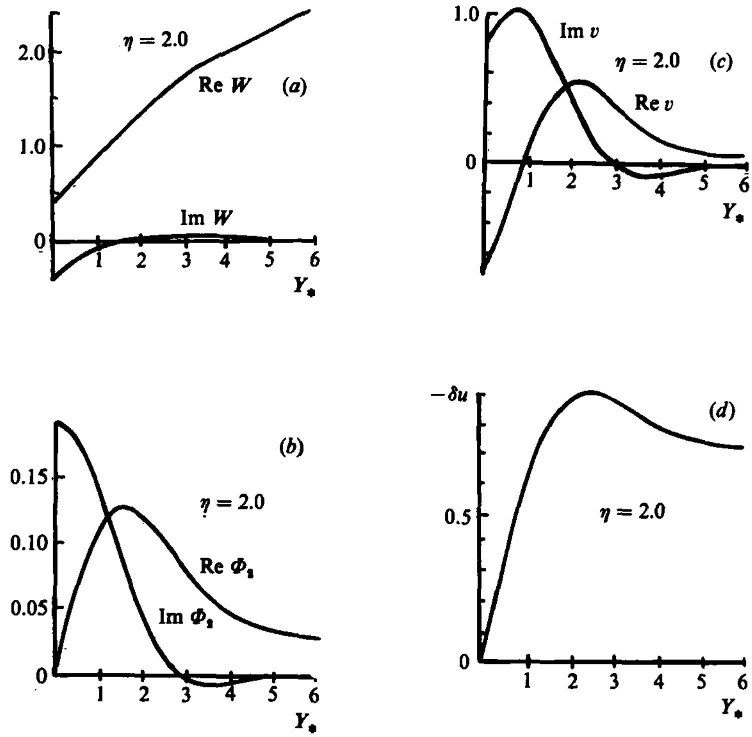

FIGURE 2. Functions of the inner problem: (*a*) the fundamental harmonic; (*b*) the second harmonic; (*c*) solution of the conjugate problem (4.17); (*d*) the zeroth harmonic.

separated from W_a. By setting the asymptotic values of W_a and its five derivatives at $\arg Y = -\frac{1}{2}\pi$, $|Y| = Y_m \gg 1$ as initial values, we can integrate (4.3) along this ray up to $Y = 0$ and thus obtain the initial conditions for further integration of this equation along the positive-Y axis. The non-homogeneous equation for Φ_2 and the conjugate equation (4.17) are solved in a similar way and the zeroth harmonic is determined from (4.10) by direct integration. The results are shown in figure 2 for Prandtl number $\eta = 2$ (here and in figure 4 $Y_* = k_0^{\frac{1}{3}} Y$).

5. The intermediate region

As mentioned in §§2 and 4, the non-stationarity and viscosity are of the same order of magnitude at $|y| \sim \kappa/\epsilon$. Introducing $Z = y/(\kappa/\epsilon)$; we can write the initial equations (2.1) and (2.2) in the form

$$Z\phi_{ZZx} - \tfrac{1}{4}\sigma_x = -\epsilon^2\kappa^{-2}[\phi_{ZZ}, \phi] - [\phi_{xx}, \phi] - \epsilon^3\kappa^{-2}\left(\frac{\partial}{\partial\tau} - \eta\frac{\partial^2}{\partial Z^2}\right)\phi_{ZZ}$$

$$+\epsilon^2\kappa^{-1}J^{(1)}\sigma_x - \epsilon^{-2}\kappa^2 Z\phi_{xxx} - \epsilon\left(\frac{\partial}{\partial\tau} - \eta\frac{\partial^2}{\partial Z^2}\right)\phi_{xx} - \epsilon^{-1}\kappa^2\eta\phi_{xxxx}, \tag{5.1}$$

$$Z\sigma_x + \phi_x = -\epsilon^2\kappa^{-2}[\sigma, \phi] - \epsilon^3\kappa^{-2}\left(\frac{\partial}{\partial\tau} - \frac{\partial^2}{\partial Z^2}\right)\sigma + \epsilon\sigma_{xx}, \tag{5.2}$$

where ψ and ρ are replaced by ϕ and σ:

$$\psi = \tfrac{1}{2}\kappa^2\epsilon^{-2}Z^2 + \phi, \quad \rho = \rho_0(0) - \kappa\epsilon^{-1}Z + \epsilon\kappa^{-1}\sigma$$

and

$$[a, b] = a_x b_Z - a_Z b_x.$$

The asymptotic expansions of the outer and inner solutions dictate the following expansions in the intermediate region:

the fundamental

$$\phi_1 = \epsilon^{\frac{1}{2}}\kappa^{\frac{1}{2}}\phi_1^{(1)} + \epsilon^{\frac{7}{2}}\kappa^{-\frac{3}{2}}\phi_1^{(2)} + \epsilon^{\frac{5}{2}}\kappa^{-\frac{1}{2}}\phi_1^{(3)} + \ldots;$$

the second harmonic

$$\phi_2 = \epsilon^3\kappa^{-1}\phi_2^{(1)} + \ldots;$$

the zeroth harmonic

$$\phi_0 = \epsilon\kappa^{\frac{1}{3}}\phi_0^{(1)} + \epsilon^3\kappa^{-1}\phi_0^{(2)} + \ldots.$$

It is easy to see that at these orders the solutions of (5.1) and (5.2) for the fundamental and the second harmonics are the trivial continuation of the outer-solution asymptotic expansions (3.11) and (3.12) through the intermediate region. This is not the case for the zeroth harmonic, for which the left-hand sides of (5.1) and (5.2) are equal to zero and the main linear term is $O(\epsilon^3\kappa^{-2})$ on the right-hand sides of each equation.

5.1. *The* $O(\epsilon\kappa^{\frac{1}{3}})$ *problem*

The evolution of the mean velocity and density perturbation is described in the intermediate region by diffusion equations:

$$\left(\frac{\partial}{\partial\tau} - \eta\frac{\partial^2}{\partial Z^2}\right)\phi_{0Z}^{(1)} = 0, \quad \left(\frac{\partial}{\partial\tau} - \frac{\partial^2}{\partial Z^2}\right)\sigma_0^{(1)} = 0. \tag{5.3}$$

The boundary conditions follow from (3.13) and (4.12):

$$\phi_{0Z}^{(1)} \to 0, \quad \sigma_0^{(1)} \to 0 \quad \text{as } Z \to \infty; \quad \phi_{0Z}^{(1)} \to |A(\tau)|^2\Phi_0(\infty), \quad \sigma_0^{(1)} \to 0 \quad \text{as } Z \to 0.$$

Thus the question of evaluating the mean velocity perturbation in the intermediate region is reduced to a simple task concerning the temperature profile in half-space, with the boundary temperature being a known function of time. The mean density distribution in this order is absent however ($\sigma_0^{(1)} = 0$).

5.2. $O(\epsilon^3\kappa^{-1})$ *problem*

$$\left.\begin{aligned}\left(\frac{\partial}{\partial\tau} - \eta\frac{\partial^2}{\partial Z^2}\right)\phi_0^{(2)} &= \frac{s}{2Z}\frac{\partial|A|^2}{\partial\tau} - \frac{5\eta+1}{8Z^3}|A|^2 s,\\ \left(\frac{\partial}{\partial\tau} - \frac{\partial^2}{\partial Z^2}\right)\sigma_0^{(2)} &= \frac{s}{Z^2}\frac{\partial|A|^2}{\partial\tau} - \frac{9}{2Z^4}|A|^2 s.\end{aligned}\right\} \tag{5.4}$$

Although it is difficult to solve (5.4), the asymptotic behaviour is readily evaluated. The first terms on the right-hand sides of (5.4) are the most important ones as $|Z| \gg 1$, but as $|Z| \ll 1$ the second terms are the most important. Consequently,

$$\phi_0^{(2)} \sim \frac{|A|^2}{2Z}s, \quad \sigma_0^{(2)} \sim \frac{|A|^2}{Z^2}s \quad \text{as } |Z| \gg 1,$$

$$\phi_0^{(2)} \sim \frac{|A|^2}{16Z}\left(5 + \frac{1}{\eta}\right)s, \quad \sigma_0^{(2)} \sim \frac{3}{4}\frac{|A|^2}{Z^2}s \quad \text{as } |Z| \ll 1.$$

Thus, the solution in the intermediate region provides the matching to both the outer and the inner solutions for the zeroth harmonic.

Note that in Paper 1 the zeroth harmonic is evaluated at $\eta = 1$ and $\Delta J = O(\nu^{\frac{1}{3}})$, i.e. on the boundary between the viscous and unsteady CL. In this case the non-stationary and the dissipative terms are already of the same order in the CL region and the intermediate region cannot be introduced. However, for matching to the outer region the asymptotic expansion of the inner solution as $Y \to \infty$ is taken in Paper 1 at fixed $|Y|/\tau^{\frac{1}{3}}$ (see (4.19) of Paper 1) rather than at fixed τ ((4.15), Paper 1) as should be done. This provides an erroneous estimate $(O(\epsilon^2 \nu^{-\frac{2}{3}}))$ of the zeroth-harmonic order in the outer region. The inner-solution matching to the outer one was not in fact made for the zeroth harmonic since the outer solution for it was not obtained.

But this result alone does not yield the incorrect estimate of $a^{(0)}$. The reason for the error is the attempt to evaluate $a^{(0)}$ from the outer solution. From the analysis presented in §4 it is clear that $a^{(0)}$ is completely determined by the *inner region* (CL) and is $O(\nu^{-1})$ as $\eta \neq 1$ apart from the order of the zeroth harmonic in the outer region.

6. Discussion

The evolution equation obtained for a specific shear-flow model (§2) is, in fact, largely a universal one. This statement is based on two facts. First, according to the Miles theorem (Miles 1961), the critical Richardson number is equal to $\frac{1}{4}$. Accordingly, for a sufficiently wide class of shear flows with an inflexion point in the velocity profile, the neutral curve $J = Ri(k)$ near its maximum, $k = k_0$, is approximately described by the formula

$$Ri(k) \approx \tfrac{1}{4} - \beta(k-k_0)^2, \quad \beta > 0. \tag{6.1}$$

In particular, in the Drazin model we have adopted, $Ri(k) = k^2(1-k^2)$ and $\beta = 2$, while in the often used Holmboe model $Ri(k) = k(1-k)$, $\beta = 1$. Secondly, the nonlinear term is entirely determined by the inner solution, i.e. it does not depend on the flow configuration as a whole. Thus, the Landau constant (4.20) can be expressed with the help of a universal function $a_0(\eta)$:

$$a = \frac{2k_0^2}{\pi}(\eta - 1)\, a_0(\eta), \tag{6.2}$$

(which, by analogy with (4.23), can also be separated into contributions of the second and zeroth harmonics $a_0(\eta) = a_0^{(2)}(\eta) + a_0^{(0)}(\eta)$).

Using (6.1), it is easy to generalize the evolution equation to the case when the z-modulation of amplitude is present. Returning in (4.19) to the original variables, taking into account (6.1) and (6.2) and introducing the modulation, we get $(\tilde{A} \equiv \epsilon A)$

$$\frac{\alpha}{k_0}\frac{\partial \tilde{A}}{\partial t} = (\tfrac{1}{4} - J)\,\tilde{A} + \beta \frac{\partial^2 \tilde{A}}{\partial x^2} + \frac{2k_0}{\pi\kappa}(\eta - 1)\, a_0(\eta) |\tilde{A}|^2 \tilde{A}. \tag{6.3}$$

Here the original flow only determines the wavenumber of the most-unstable mode k_0 and the positive coefficients α and β. In the Drazin model, $k_0 = 1/\surd 2$, $\alpha = 1$, and $\beta = 2$, while in the Holmboe model $k_0 = \frac{1}{2}$, $\alpha = \frac{1}{2}$, and $\beta = 1$.

The evaluation of the integral in (4.20) in the interval $\frac{1}{8} < \eta < 8$ has shown that both contributions to the Landau constant have the same sign but $a_0^{(0)}$ is larger than $a_0^{(2)}$ by a factor of approximately 5, i.e. the nonlinear terms in (6.3) are determined mainly by the interaction of the fundamental with the mean velocity and density distortions (the zeroth harmonic). $I^{(2)} = (1-\eta)\, a_0^{(2)}(\eta)$ and $I^{(0)} = (1-\eta)\, a_0^{(0)}(\eta)$ are plotted in figure 3.

When the Prandtl number $\eta < 1$, nonlinearity limits the instability at the level

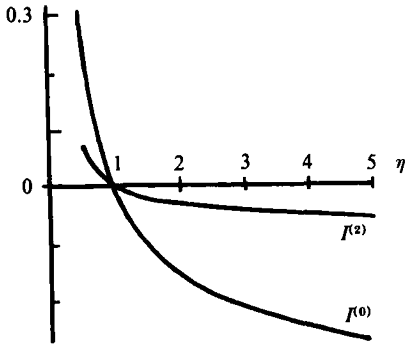

FIGURE 3. Evaluation of the integral in (4.20): $I^{(2)} = (1-\eta)\, a_0^2(\eta)$; $I^{(0)} = (1-\eta)\, a_0^{(0)}(\eta)$.

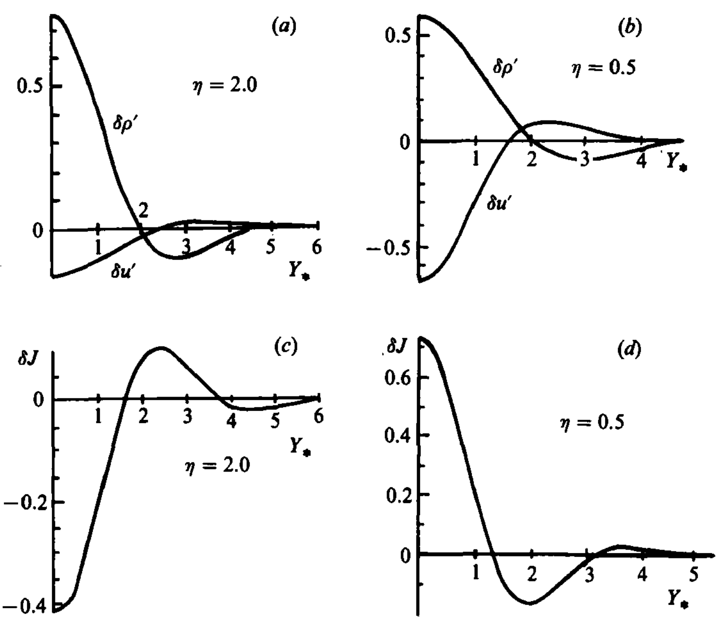

FIGURE 4. Perturbations of the mean flow parameters at different values of the Prandtl number: $\eta = 0.5$; $\eta = 2$ inside the CL. (a, b) Perturbation of the gradients of mean velocity u' and density ρ'; (c, d) variation in Richardson number (in units ϵ^2/κ).

$\tilde{A}_0 \sim [(\frac{1}{4}-J)\,\kappa/(1-\eta)]^{\frac{1}{2}}$. If $\eta > 1$, nonlinearity plays a destabilizing role and the exponential growth of amplitude at $\tilde{A} \sim \tilde{A}_0$ is replaced by power-law growth, according to the law $\tilde{A} \sim (t_0-t)^{-\frac{1}{2}}$ which describes the so-called 'burst' instability and will be valid unless at $\tilde{A} \sim \nu^{\frac{1}{2}}$ the CL becomes nonlinear.

One can give the following interpretation to the change in nonlinear stability. According to the Le Chatelier principle, the mean velocity and density (figure 4) are perturbed so as to decrease the velocity and density gradients of the original flow, i.e. $\delta u' < 0$, $\delta \rho' > 0$. The variation of Richardson number which governs the flow stability,

$$\delta J = -\frac{1}{4}\left(2\frac{\delta u'}{u_c'} + \frac{\delta \rho'}{|\rho_0'|}\right),$$

is determined by competition between $\delta u'$ and $\delta\rho'$. If thermoconductivity exceeds the viscosity ($\eta < 1$), the density perturbation diffuses more intensively than the velocity perturbation does, so that $\delta\rho'$ is small and $\delta J > 0$, which means stabilization, but if, on the contrary, the viscosity dominates ($\eta = 1$), the velocity perburbation diffuses faster, so that $\delta J < 0$ (destabilization). Consequently, at some $\eta = \eta_0$, the Landau constant must vanish. In the high-Reynolds-number limit $\eta_0 = 1$ (see (4.21)).

In §§1 and 5 we have already discussed the results of Paper 1. We believe that this paper contains the right estimate ($O(\nu^{-1})$) of the second harmonic contribution $a^{(2)}$ to the Landau constant and demonstrates that the Landau constant changes sign at $\eta = \eta_0 \approx 1$. This conclusion is verified by means of a numerical solution of (2.1) and (2.2) at relatively low Reynolds numbers ($\nu^{-1} < 200$) and at Prandtl numbers near unity ($0.7 < \eta < 1.3$).

The error in Paper 1 in estimating the zeroth-harmonic contribution has been discussed above. Paper 1 also contains an incorrect estimate of the Landau constant ($O(\nu^{-\frac{1}{3}})$) as $\eta = 1$. The reason for this error is again that the outer solution (for the second harmonic in this case) has not been obtained and this leads to the omission of some terms in the inner solution (see Appendix B). The correct result is $a_2 = O(\nu^{-\frac{1}{2}})$ (see (4.22)). Nevertheless this fact does not change the form of the evolution equation at $\eta = 1$ (see (1.6) and (7.5) of Paper 1) because the nonlinear term of the fifth order ($a_4|A|^4A$, $a_4 = O(\nu^{-2})$) dominates as $\Delta J > \nu$.

The affirmation contained in Paper 1 that the Landau constant at $\eta = 1$ tends to zero as $\nu \rightarrow 0$ is also questionable. It seems to us that this conclusion is based on the fact that the point where the $a(\eta)$-graph crosses the η-axis tends to $\eta = 1$ as $\nu \rightarrow 0$ and that at $\nu^{-1} = 200$ $|a(1)|$ is less than its value at $\nu^{-1} = 100$ (see figure 2(a, b) of Paper 1); this is evidently due to the relatively low values of the Reynolds number. We believe that, as $\nu \rightarrow 0$, the Prandtl number η_0 at which $a(\eta)$ vanishes tends to unity (see (4.21)): $\eta_0 - 1 = O(\nu^{\frac{1}{2}})$ but at the same time the slope of the graph increases: $\partial a/\partial\eta = O(\nu^{-1})$ and these result in the Landau constant increasing: $a(1) = O(\nu^{-\frac{1}{2}})$.

Another point concerning the nonlinear evolution in the viscous CL regime is that for a correct formulation of the problem, the linear time of growth γ_L^{-1}, and the nonlinear time $(\tilde{A}^2 a)^{-1}$, equal to it in the case of interest, must be less than the time for the original flow to spread due to viscosity ν^{-1}: $\gamma_L \sim a\tilde{A}^2 \gg \nu$ or $\tilde{A} \gg (\nu/a)^{\frac{1}{2}}$. On the other hand, the viscous CL regime is realized when the amplitude is not too large: $\tilde{A} \ll \nu^{1/(3p)}$. In our case, $a \sim \nu^{-1}$ and $p = \frac{2}{3}$; therefore these conditions are compatible in the region of the parameters $\Delta J > \nu$, $\nu < \tilde{A} < \nu^{\frac{1}{2}}$ (see figure 1). In an analogous problem for a uniform flow, these conditions cannot be compatible: for such a flow Schade (1964) has obtained $a = O(1)$, $p = \frac{1}{2}$ which leads to discrepant inequalities: $\tilde{A} \gg \nu^{\frac{1}{2}}$ and $\tilde{A} \ll \nu^{\frac{2}{3}}$. To avoid the discrepancy, one has to introduce an artificial force field (Huerre 1980) to maintain the original flow. With this formulation of the problem, our correct account of the zeroth harmonic gives $a = O(\nu^{-\frac{1}{2}})$ but not $O(1)$ as in the papers cited. Let us emphasize that for a stratified flow the nonlinearity is stronger and there is no need to introduce the forces. The above nonlinear theory is restricted to small enough amplitudes ($\tilde{A} \ll \nu^{\frac{1}{2}}$) and supercriticalities ($\Delta J < \nu^{\frac{1}{3}}$) and, therefore, as the next step it is interesting to trace the evolution in the nonlinear and unsteady CL regimes (regions II and III of the parameters in figure 1). This issue is insufficiently studied not only for stratified flow but also for other flows. We want to draw the readers' attention to the excellent paper by Reutov (1982) who has investigated for a plane Poisseuille flow the transition from viscous to nonlinear CL regime.

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Appendix A. Why does the indentation rule not work in the nonlinear theory?

Extension into a complex plane is one of the powerful tools of analysis and has wide application in seeking the solutions of differential equations, evaluating the integrals, and in other tasks. But its correct use requires investigating the analytic properties of the functions used.

The set of equations (2.1), (2.2) on a complex-y plane has no singularities in the circle $|y| < 1$; consequently, its true solution is analytic in this area. But a compound solution (outer+inner) approximating it does not possess this property. Let us consider in more detail the question of the analytic properties of a compound solution because we use it in all calculations.

The algorithms for constructing the outer and inner solutions are, in fact, identical and represent step-by-step iterative calculations of the functions $\phi^{(i)}$ which are the solutions of equations

$$\mathscr{L}\phi^{(i)} = \mathscr{R}^{(i)}. \tag{A 1}$$

In the outer problem, the role of operator $\mathscr{L}$ is played by the Taylor–Goldstein operator but, in the inner problem, by the operator

$$\mathscr{L} = \eta\mathscr{D}^6 - \mathrm{i}(\eta Z\mathscr{D}^4 + \mathscr{D}^2 Z\mathscr{D}^2) - Z^2\mathscr{D}^2 - \tfrac{1}{4}, \quad \mathscr{D} = \frac{\partial}{\partial Z}, \tag{A 2}$$

to which, using the replacement $Z = (nk_0)^{\frac{1}{3}}Y$, all operators $\mathscr{L}_n$ at $n \neq 0$ may be reduced. The right-hand sides are constructed from preceding iterations ($\phi^{(m)}$, $m < i$) but the first iteration is the solution of the homogeneous equation

$$\mathscr{L}\phi^{(1)} = 0. \tag{A 3}$$

Therefore, the analytic properties of $\phi^{(i)}$ are determined by those of the operator $\mathscr{L}$ and by preceding iterations, i.e. finally, by the properties of problem (A 3).

The Taylor–Goldstein operator has singular point $y = 0$, which is the branching point for solutions of (A 3). The asymptotic behaviours of these solutions as $y \to +0$ are

$$\phi_a \sim y^{\frac{1}{2}}, \quad \phi_b \sim -y^{\frac{1}{2}} \ln(\tfrac{1}{2}y). \tag{A 4}$$

Operator (A 2) has no singular points, and therefore the solutions of (A 3) in the inner region are holomorphic functions. Two of them as $Z \to \infty$ are of the 'power' type ($\Phi_{1,2}$) and four solutions ($\Phi_3, \ldots, \Phi_6$) are of 'WKB' type. If $\eta \neq 1$

$$\Phi_1 \sim Z^{\frac{1}{2}}, \quad \Phi_2 \sim -Z^{\frac{1}{2}} \ln(\tfrac{1}{2}Z), \quad \Phi_{3,4} \sim Z^{-\frac{5}{4}} \mathrm{e}^{\pm S_1(Z)},$$

$$\Phi_{5,6} \sim Z^{-\frac{3}{2}} \mathrm{e}^{\pm S_2(Z)}, \quad S_1 = \tfrac{\sqrt{2}}{3}(1+\mathrm{i})Z^{\frac{3}{2}}, \quad S_2 = \left(\frac{2}{\eta}\right)^{\frac{1}{2}} \frac{1+\mathrm{i}}{3} Z^{\frac{3}{2}}. \tag{A 5}$$

Each particular solution of (A 3) is a superposition of Φ_n, and its asymptotic representation changes when we pass the Stokes lines (see, for example, Fedoryuk 1983) $\mathrm{Re}\, S_1 = \mathrm{Re}\, S_2 = 0$ on which the asymptotic values of all Φ_n are 'equal-valued', i.e. they increase or decrease not faster than some power of Z. Operator (A 2) has three Stokes lines: $\arg Z = -\frac{7}{6}\pi$, $\arg Z = -\frac{1}{2}\pi$, and $\arg Z = \frac{1}{6}\pi$.

If we take on the Stokes line solution having a 'pure' asymptotic behaviour, say,

Φ_1, then as we move away from the Stokes line, the exponentially small terms will add to it and after passing through the next Stokes line these terms will become exponentially growing.

Thus, the asymptotic representation may remain unchanged only in the region that contains no more than one Stokes line. On this basis, one can construct uniquely the solutions, which have as $Z \to \infty$ asymptotic behaviour of the kind (A 4) at both ends of the real axis. Let Φ_a and Φ_b on the Stokes line $\arg Z = -\frac{1}{2}\pi$ as $Z \to \infty$ have asymptotic representations

$$\Phi_a = Z^{\frac{1}{2}}(1+O(Z^{-3})), \quad \Phi_b = -Z^{\frac{1}{2}} \ln(\tfrac{1}{2}Z)(1+O(Z^{-3})); \tag{A 6}$$

then they will have the same representations (plus exponentially small additions composed of $\Phi_3, \ldots, \Phi_6$ in sector $-\frac{7}{6}\pi < \arg Z < \frac{1}{6}\pi$ ($\mathscr{M}$ region)). In the remaining sector $\frac{1}{6}\pi < \arg Z < \frac{5}{6}\pi$ Φ_a and Φ_b are exponentially increasing.

By analytically continuing ϕ_a and ϕ_b from the positive semiaxis to $\mathscr{M}$, we see that asymptotic expansions of ϕ_a and Φ_a, ϕ_b and Φ_b are matched everywhere in $\mathscr{M}$ and the compound function obtained provides a uniformly valid approximation of the fundamental harmonic of a true solution in sector $\mathscr{M}$ of the circle $|y| < 1$.

Now let Φ be a certain term of an inner-solution expansion. The function Φ can be matched to its respective fragment ϕ of the outer solution only in that sector $\mathscr{M}_1$ of the complex plane where the outer asymptotic representation of Φ is a power law since the outer solution has only such a behaviour (see (3.11)–(3.13)). In other words, the compound function in this sector serves as a uniformly valid term of the true solution. We shall refer to region $\mathscr{M}$ as the compound's analyticity region or, more often, as the analyticity region of its inner component. In the same sense, this term is used in the main text. Thus, Φ_a and Φ_b are analytic in $\mathscr{M}$. Let us stress that it is impossible to construct the solution of (A 3) that is analytic in the upper half-plane as it contains two Stokes lines, and growing exponential terms will inevitably arise in the inner-solution's asymptotic expansion.

Returning to the generalized equation (A 1), one may formulate the main result as follows: the solution $\phi^{(i)}$ is analytic in the region $\mathscr{N}$, if (*a*) $\mathscr{R}^{(i)}$ is analytic in $\mathscr{N}$, (*b*) $\mathscr{N}$ contains not more than one Stokes line of the operator $\mathscr{L}$.

Consider now the sequence of analytic properties of the (A 3) problem.

(i) One may displace the integration contour into the lower half-plane of complex-y only when the integrand is analytic in $\mathscr{M}$. Particularly, in the linearized problem, the $\mathscr{R}^{(i)}$ are linear functions of $\phi^{(1)}$ and of its derivatives, i.e. they are analytic in $\mathscr{M}$; consequently, all $\phi^{(i)}$ are analytic in $\mathscr{M}$. Hence, both aspects of Lin's indentation rule from below follow from this (see §1). Thus, the linear part of the evolution equation may be obtained by the usual method of indenting the contour below the singular point.

(ii) The main point of the inner problem is to determine the asymptotic behaviour of the solution $\Psi_n^{(i)}$ of (A 1)-type equations. The general form of $\Psi_n^{(i)}$ asymptotic expansions can be determined on the basis of the right-hand side asymptotic representations $\mathscr{R}^{(i)\pm}$:

$$\Psi_n^{(i)} \sim f^{\pm} + (m^{\pm} + n^{\pm} \ln \tfrac{1}{2}Z) Z^{\frac{1}{2}}(1+O(Z^{-3})).$$

The $f^{\pm}$ are fully reconstructed by means of $\mathscr{R}^{(i)\pm}$ but $m^{\pm}$ and $n^{\pm}$ remain undetermined since the solution of (A 1) is defined up to the general solution of the homogeneous equation. If the right-hand side of (A 1) is analytic in the lower half-plane, as in the case with (4.7) and (4.13), a solution exists that is analytic in $\mathscr{M}$ (the Stokes-lines configuration is such that the analyticity in the lower half-plane means analyticity in $\mathscr{M}$). This solution has a uniform asymptotic representation in $\mathscr{M}$; consequently, the m and n coefficients are constant in $\mathscr{M}$, i.e. $m^+ = m^-$ and $n^+ = n^-$.

(iii) In nonlinear problems, the right-hand side of (A 1), $\mathscr{R}^{(i)}$, contains, together with W_a, W_b and their derivatives which are analytic in $\mathscr{M}$, plus the complex-conjugate functions $\overline{W}_a$ and $\overline{W}_b$ which are non-analytic in the $-\frac{5}{6}\pi < \arg Y < -\frac{1}{6}\pi$ sector (e.g. the $|A|^2 A\mathscr{R}_N$ term on the right-hand side of (4.14)). In this case, evaluating the $\Psi_n^{(i)}$ asymptotic expansions requires integrating the appropriate equation along the real axis (or along any other curve which lies entirely in the branch of analyticity, i.e. in the $|\arg Y| < \frac{1}{6}\pi$ and $-\frac{7}{6}\pi < Y < -\frac{5}{6}\pi$ sectors, and connects $+\infty$ and $-\infty$) and, in general, $m^+ \neq m^-$, $n^+ \neq n^-$ as in (4.15).

We should emphasize that in nonlinear problems the integrands (in solvability conditions) inevitably contain $\overline{W}_a$ together with W_a and are non-analytic in the lower half-plane. Therefore, the 'indentation' rule is invalid for nonlinear problems.

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